The area-entropy relation for black holes

Alexios P. Polychronakos

Institute for Theoretical Physics, Uppsala University, Sweden
and Physics Department, University of Ioannina, Greece

Abstract

In several quantum gravity models, black holes are treated as composite objects consisting of many identical components. Under the sole assumption that the area of the black hole is a quantity additive in the components, we demonstrate that the Beckenstein-Hawking area-entropy relation is recovered, provided that the components obey *distinguishable quantum statistics*. Some justification on why it would be so in known models is given. This also imples that the area of a black hole has large local fluctuations which violate no know astrophysical data but may have testable implications. This talk is based on work done in collaboration with A. Alekseev and M. Smedbäk reported in hep-th/0004036 [1].

In the framework of Quantum Gravity we treat black holes as quantum objects. As such, they are characterized by quantum numbers: mass, electric charge and angular momentum. For Schwarzschild black holes (neutral, with angular momentum zero) the only quantum number which is left is the mass M. It is related to the area A of the black hole horizon by formula

$$A = \frac{16\pi G^2 M^2}{c^4},$$

where G is the gravitational constant. Important questions in black hole physics are first, what the spectrum of A looks like and, second, what the degeneracies of states are for a given value of A.

In the absence of a definitive Quantum Gravity theory, the answers to these questions depend on the model of a black hole. In several approaches one assumes that a black hole consists of elementary objects contributing additively to its area. For instance, in the Ashtekar's approach (see *e.g.* [2, 3]) these are Wilson lines of the Ashtekar's connection A^a_μ. In the M-theory approach (see, e.g., [8]) these are D_0-branes. Each of the elementary objects can be in a number of states which we enumerate by an integer n. An elementary object in the state n gives a contribution A_n to the total area, and has a degeneracy $g(n)$. The eigenvalues of the area operator then acquire the form

$$A = \sum_{n=1}^{\infty} N_n A_n, \tag{1}$$

where N_n is the number of elementary objects in the state n. The multiplicity of states with area A is $\Omega(A)$ and its asymptotic when A is macroscopically large defines the entropy of the black hole

$$S(A) = \ln \Omega(A).$$

The spectrum of the area A and the behaviour of the entropy $S(A)$ depend on the spectrum of the elementary area A_n, the elementary multiplicities $g(n)$ and the degree of distinguishability of the components. For the case of Ashtekar gravity [4] the index n can be associated with a spin j of an $SU(2)$ representation. Then, j takes integer and half-integer values and according to [4]

$$A_j = \gamma\sqrt{j(j+1)}. \tag{2}$$

with γ a coefficient of the order of the Planck area A_{pl}. This result is contained as a subset of a more general result [5]. An alternative spectrum is [1]

$$A_j = \gamma(j + \frac{1}{2}). \tag{3}$$

In particular, this implies that the spectrum of the area operator is equidistant. Such a situation was first considered by Bekenstein and Mukhanov [6]. This spectrum implies certain specific properties of the black hole radiation:

There exists an energy quantum

$$\hbar\omega_0 = \frac{\gamma c^2 l_p^2}{8G^2 M},$$

Energy can be radiated only in integer multiples of this energy quantum and the frequency distribution will be that of Hawking radiation.

In what follows we shall consider the issue of black hole entropy. It will be seen that the *area law* $S \propto A$ is to a large extent generic. A more subtle issue is what the statistically preferrable state of the system is. The analysis of [7] in the case of Ashtekar gravity and of [8] showed that the most probable configuration has all the elementary objects in the same spin multiplet. This can be viewed as Bose condensation. We show instead that for completely distinguishable elementary objects with $g(n) = n = 2j + 1$, the degeneracy for the $SU(2)$ multiplet with spin j, one obtains a Gibbs distribution for the N_n's with many of them nonvanishing in the most probable configuration. Our result implies that the area operator has strong fluctuations, $\Delta A \sim \sqrt{A}$, whereas in the case of Bose condensation the fluctuation of area would have been strongly suppressed.

In Asthekar's gravity approach, the surface area of the black hole horizon is given by several small edge intersections with the surface, each of them contributing additively to the total area [7]. The number of edges intersecting the black hole surface can vary. The edges are therefore a concrete realization of the component picture of a black hole. The number of components can vary, and thus they can be considered as the constituents of a grand canonical system. Standard thermodynamics, then, determines the entropy S of the surface system, which in turn determines the entropy of the entire black hole, to be given by

$$S = k(-A \ln z + \ln Z), \tag{4}$$

where k is the Boltzmann constant, A is the conserved black hole area, β is temperature-like parameter dual to the area and Z is the partition function,

$$Z = \sum_A \Omega(A) e^{-\beta A} = \sum_{A,N} \Omega(A, N) e^{-\beta A}. \tag{5}$$

$\Omega(A)$ is the multiplicity of states of area A, and $\Omega(A,N)$ is the multiplicity of state of area A and N components. The summation is over all possible areas and number of components.

To proceed, we will consider the components as identical but *completely distinguishable* independent quantum systems. The partition function becomes

$$Z = \sum_N Z_1^N = \frac{1}{1-Z_1} \tag{6}$$

where Z_1 is the partition function of each component

$$Z_1 = \sum_A \Omega(A,1) z^A. \tag{7}$$

The above expression for Z implies that β can never be less than the *Hagedorn temperature* parameter β_o, fixed by the relation $Z_1(\beta_o) = 1$.

From equation (5), the area is related to the partition function by $A = -\frac{d\ln Z}{d\beta} = -\frac{1}{1-Z_1} \cdot \frac{dZ_1}{d\beta}$. Restricting our considerations to macroscopic areas $A \gg 1$ only, this relation implies that $1 - Z_1 \sim \frac{1}{A}$, and thus $\beta \to \beta_o$ and $\ln Z$ grows only as $\ln A$. The dominant contribution to the entropy will for therefore be given by $S = k\beta_o A$. The entropy is always proportioal to the area. This is a completely general result, valid for any system consisting of distinguishable components.

To fix the proportionality constant, we restrict our considerations to the case of edges with the area spectrum given by equation (3), i.e. $A = n$, (measured in units of $4\pi\gamma l_p^2$) $n = 1, 2, 3, ...$, and the degeneracy function $\Omega(A,1) = \Omega(n,1) = g(n) = n$, where g(n) is the degeneracy of an edge of area level n, corresponding to area n. Note that the states enumerated by the spin quantum number j, taking integer or half-integer values, are now enumerated by $n = 2j+1$, a positive integer. Putting $z = e^\beta$ the quantity Z_1 takes the form

$$Z_1 = \frac{z}{(1-z)^2}. \tag{8}$$

The partition function diverges at the Hagedorn temperature $z = z_0 = \frac{3-\sqrt{5}}{2}$ and he area becomes macroscopic as $\beta \to -\ln z_0$. In that limit the entropy becomes

$$S = \frac{kA}{4\pi\gamma l_p^2} \ln\left(\frac{3+\sqrt{5}}{2}\right), \tag{9}$$

where we have acknowledged that the area previously was measured in units of $4\pi\gamma l_p^2$ by reinserting this factor.

Equation (9) represents our final result for the entropy of a black hole. The validity of this derivation depends crucially on an important claims: that the area constituents, i.e. the edges in the Ashtekar's gravity approach, are completely distinguishable. Let us attempt to justify this claim.

The edges in the Ashtekar's gravity approach could *a priori* be either *identical*, *partially distinguishable* or *completely distinguishable*. Completely distinguishable edges would mean that: (1) There is a difference between assigning two given spin values j_1 and j_2 to two different edges according to (edge 1, edge 2)= (j_1, j_2) and (edge 1, edge 2)= (j_2, j_1). (2) Even if $j_1 = j_2$, there is a difference between assigning two different spin states m_1 ans m_2 to the two different edges.

In calculating the entropy of equation (9), the edges were considered to be *completely distinguishable*, i.e. both claims (1) and (2) are imposed. Since the positions of different edges are determined by the *spin network*, which is formed by the way in which the edges are connected to each other, this viewpoint appears to be the one best describinging the physical model, and as such, the one we should adopt.

Ashtekar et al. [7] adopt a different view: The edges are *partially distinguishable*. Indeed, the multiplicity formula $\Omega(\{N_n\}) = \prod_n g(n)^{N_n}$ applies if claim (1) above is *not* adopted while still keeping (2). Applying (5) with this counting of states we obtain

$$Z = \sum_{\{N_n\}} \Omega(\{N_n\}) e^{-\beta A} = \prod_n \frac{1}{1 - g(n) e^{-\beta A(n)}} \tag{10}$$

If each edge does not have an exponentially increasing density of states (in which case it would be itself a macroscopic black hole) the only poles of the above expression are at $e^{\beta A(n)} = g(n)$. Calling $\bar{\beta}_o$ the lowest of these values of β, occuring for some n_o, we deduce that for macroscopic areas the model will exhibit Bose condensation at the spin $j_o = (n_o - 1)/2$ and the entropy will be $S = \bar{\beta}_o A$.

$$Z = \sum_A \Omega(A) e^{-\beta A} = \sum_{A,N} \Omega(A, N) e^{-\beta A}. \tag{11}$$

For our area function (3) and the degeneracy function $g(n) = n$ this gives an entropy $S = \frac{kA\gamma_0}{4l_p^2}$ with $\gamma_0 = \frac{\ln 3}{3\pi}$. Bose condensation occurs at spin $j = 1$. For Ashtekar's choice of area function the result is instead $\gamma_0 = \frac{\ln 2}{\pi\sqrt{3}}$, and Bose condensation occurs at $j = \frac{1}{2}$.

Finally, if the edges are *identical* we adopt neither claim (1) nor (2) and have the equivalent of a Bose-Einstein gas. Then, it turns out that the entropy is related to the area by $S \propto A^t$, where the exponent satisfies $t < 1$ if each edge does not have an exponential density of states. For the choices of area spectrum (3) and degeneracy function $g(n) = n$, the exponent acquires the value $t = \frac{2}{3}$. No Bose condensation occurs.

In conclusion, the area law $S = \beta_o A$ and the appearence of a Hagedorn temperature are the main results. These are extremely generic, requiring only *some* distinguishability of the elementary components. Since the parameter $\gamma$is not fixed by the quantum theory [**?**], the different values of β_o are of somewhat secondary importance. There is, however, a crucial physical difference between our result and the result of Ashtekar et al. In our fully distinguishable case no Bose condentation occurs and the area within any solid angle of the black hole will exhibit fluctuations of order $\sqrt{A}$, while in the Ashtekar et al. result Bose condensation occurs and area fluctuations are strongly suppressed. This, in connection with the different area spectrum for low spin values, is expected to have observable physical coonsequences.

Are there any gross violations of astrophysical data due to these fluctuations? We can show that this is not the case. The area of a macroscopic black hole contained in some solid angle Ω as seen by an asymptotic observer will have fluctuations of the order

$$\Delta A \approx \sqrt{\Omega A A_{pl}} \tag{12}$$

and thus the relative fluctuations are

$$\frac{\Delta A}{A} \sim \sqrt{\frac{A_{pl}}{A}} \ll 1 \tag{13}$$

In fact, given that $A \sim R^2$, with R the radius of the black hole horizon, the above relation implies

$$\Delta R \sim \ell_{pl} \tag{14}$$

with ℓ_{pl} the Planck length. This is reasonable, and demonstrates that any effect sensitive to the area fluctuations that we are predicting would need to probe the sizeo fthe black hole down to the Planck scale. Only some very high-energy astroparticle processes could conceivably achieve that.

References

[1] A.Yu.Alekseev, A.P.Polychronakos and M.Smedbäck, hep-th/0004036, *On Area and Entropy of a Black Hole*

[2] R.Gambini, J.Pullin, Cambridge University Press, Cambridge, 1996, *Loops, Knots, Gauge Theories and Quantum Gravity*

[3] Abhay Ashtekar, Advanced Series in Astrophysics and Cosmology, Vol.6, World Scientific, Singapore, 1991, *Lectures on non-perturbative canonical gravity*

[4] Carlo Rovelli, Lee Smolin, Nucl.Phys. B442 (1995) 593-622, gr-qc 9411005, *Discreteness of Area and Volume in Quantum Gravity*

[5] Simonetta Frittelli, Luis Lehner and Carlo Rovelli, Class.Quant.Grav. 13 (1996) 2921-2932, gr-qc 9608043 v2 *The Complete Spectrum of the Area from Recoupling Theory in Loop Quantum Gravity*

[6] J.D. Bekenstein, V.F. Mukhanov, Phys.Lett. B360 (1995) 7-12, gr-qc 9505012, *Spectroscopy of the Quantum Black Hole*

[7] A.Ashtekar, J.Baez, A.Corichi, K.Krasnov, Phys.Rev.Lett. 80 (1998) 904-907, gr-qc 9710007, *Quantum Geometry and Black Hole Entropy*

[8] S.Chaudhuri, D.Minic, Phys.Lett B433 (1998) 301-306, hep-th 9803120, *On the Hagedorn Transition and Collective Dynamics of D0-branes*